

# Stability of condensate in superconductors

P. Lipavský<sup>1,2</sup>, B. Šopík<sup>1,2</sup>, M. Männel<sup>3</sup> and K. Morawetz<sup>4,5</sup>

<sup>1</sup>*Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 12116 Prague 2, Czech Republic*

<sup>2</sup>*Institute of Physics, Academy of Sciences, Cukrovarnická 10, 16253 Prague 6, Czech Republic*

<sup>3</sup>*Institute of Physics, Chemnitz University of Technology, 09107 Chemnitz, Germany*

<sup>4</sup>*International Center of Condensed Matter Physics,  
University of Brasilia, 70904-970, Brasilia-DF, Brazil and*

<sup>5</sup>*Münster University of Applied Science, Stegerwaldstrasse 39, 48565 Steinfurt, Germany*

According to the BCS theory the superconducting condensate develops in a single quantum mode and no Cooper pairs out of the condensate are assumed. Here we discuss a mechanism by which the successful mode inhibits condensation in neighboring modes and suppresses a creation of non-condensed Cooper pairs. It is shown that condensed and noncondensed Cooper pairs are separated by an energy gap which is smaller than the superconducting gap but large enough to prevent nucleation in all other modes and to eliminate effects of noncondensed Cooper pairs on properties of superconductors. Our result thus justifies basic assumptions of the BCS theory and confirms that the BCS condensate is stable with respect to two-particle excitations.

PACS numbers: 71.10.-w, 74.20.-z, 03.75.Ss, 05.30.Fk

## I. INTRODUCTION

Seven decades ago London put forward the idea to explain superconductivity by a rigid quantum wave function covering all superconducting electrons. His bold vision is included in all recent theories would the role of the wave function be fulfilled by the Ginzburg-Landau (GL) complex order parameter or the Bardeen-Cooper-Schrieffer (BCS) gap function<sup>1,2</sup>. The superconducting condensate is not called rigid but it is assumed to be sufficiently stable to form an effective vacuum of a new state, the state of broken symmetry<sup>3</sup>. This stability is always used but its origin is not yet clear.

In superconductors the macroscopic wave function is a phenomenological theoretical tool with a complicated underlying microscopic picture. In contrast, the macroscopic wave function of dilute superfluid gases is identical to the intuitively clear Schrödinger wave function of the lowest energy single-particle state macroscopically occupied due to the Bose-Einstein condensation.<sup>4</sup> On superfluids we can outline the problem addressed here for superconductors.

Landau has shown that the supercurrent is halted if external perturbations excite bosons from the condensate to neighboring low-lying states.<sup>5</sup> According to this Landau criterion a supercurrent in the ideal (noninteracting) Bose gas is unstable. Indeed, in the ideal Bose gas the energy of a neighbor state differs by the kinetic energy which is quadratic in momentum of this excitation. For any slow perturbation one then finds states of slower velocity vulnerable to excitation by Cherenkov-like mechanism. In real Bose systems the situation is different. The interaction between bosons causes a reconstruction of the energy spectrum from the quadratic to the linear form of acoustic type. Perturbations slower than the corresponding sound velocity then cannot excite particles so that the supercurrent is stable and the condensate wave function reveals the London rigidity.

Excitations of the superconductors are principally different. According to the BCS theory well supported by experimental experience, see e.g. Ref. 6, the superconductivity is controlled by fermionic quasiparticles resulting from broken Cooper pairs. We don't discuss here this familiar mechanism. Our central question is: *Why Cooper pairs of nonzero momenta are not excited from the BCS condensate?*

Here we show that the multiple scattering corrections to the T-matrix<sup>7</sup> lead to gaps in the single-particle and two-particle energy spectra. The single-particle gap is the familiar BCS gap which is known to guarantee stability with respect to the excitation of fermionic quasiparticles. We focus on the two-particle gap and show that it guarantees stability with respect to nucleation of superconducting condensate in two or more momentum states and with respect to excitation of noncondensed Cooper pairs. These two gaps thus imply the stable condensate.

The paper is organized as follows. In Sec. II we introduce the T-matrix approach first on a general level, Sec. II A, and then its simplified form for the separable interaction of the BCS type, Sec. II B. Then we evaluate the T-matrix near the critical temperature for the condensate in Sec. II C, and for the noncondensed Cooper pairs in Sec. II D. In Sec. III we discuss the stability of the condensate with respect to nucleation of a second condensate, Sec. III A, and with respect to excitation of noncondensed Cooper pairs, Sec. III B. Comparing the present approximation with the Kadanoff-Martin theory in Sec. III C we point out the role of multiple scattering correction in formation of the gap in the two-particle energy spectrum. Section IV is a summary.

## II. T-MATRIX APPROACH

The noncondensed Cooper pairs are not covered by the BCS or Eliashberg theory. These approaches treat

Cooper pairs within the mean field which is nonzero only for states with macroscopic bosonic occupation – the states with the condensate. In our study we employ the T-matrix in which the pairing is independent of the condensation. We use the Galitskii-Feynman approximation with multiple scattering corrections<sup>7</sup> in modification of Ref. 8.

### A. General equations

Let us introduce the theory. The full propagator is given by the Dyson equation

$$G_{\uparrow} = G^0 + G^0 \Sigma_{\uparrow} G_{\uparrow}, \quad (1)$$

where the selfenergy

$$\Sigma_{\uparrow} = \frac{k_B T}{\Omega} \sum_Q \sigma_{Q\uparrow} \quad (2)$$

is a sum over four-momentum  $Q \equiv (\omega, \mathbf{Q})$  of interacting pairs, with Matsubara's frequencies  $\omega$  and discrete wave vectors  $\mathbf{Q}$  corresponding to the sample volume  $\Omega$ . In the case of condensation mode  $Q$  is a four-momentum of a Cooper pair.  $\sigma_{Q\uparrow}$  we call a  $Q$ -part.

To avoid double-counts, the internal lines of the  $Q$ -part of the selfenergy should not include processes related to the  $Q$ -mode itself. To this end we introduce the  $Q$ -reduced propagator

$$G_{Q\downarrow} = G^0 + G^0 \Sigma_{Q\downarrow} G_{Q\downarrow}, \quad (3)$$

which is dressed by all but the  $Q$ -part of the selfenergy

$$\Sigma_{Q\downarrow} = \frac{k_B T}{\Omega} \sum_{Q' \neq Q} \sigma_{Q'\downarrow}. \quad (4)$$

The  $Q$ -part of the  $\uparrow$  selfenergy is obtained by closing the loop of  $\downarrow$  line of the T-matrix by the reduced propagator

$$\sigma_{Q\uparrow}(k) = \mathcal{T}_{\uparrow\downarrow}(k, Q-k; k, Q-k) G_{Q\downarrow}(Q-k). \quad (5)$$

The T-matrix is constructed from the  $Q$ -reduced propagators in the  $\downarrow$  line and full propagators in the  $\uparrow$  line

$$\begin{aligned} \mathcal{T}_{\uparrow\downarrow}(k, Q-k; p, Q-p) &= D(k, Q-k; p, Q-p) \\ &- \frac{k_B T}{\Omega} \sum_{k'} D(k, Q-k; k', Q-k') G_{\uparrow}(k') G_{Q\downarrow}(Q-k') \\ &\times \mathcal{T}_{\uparrow\downarrow}(k', Q-k'; p, Q-p). \end{aligned} \quad (6)$$

Except for the reduced propagator, this is the standard ladder approximation. Here  $D$  can be either a general phonon propagator with vortices included or an effective interaction potential of the BCS type. The set of equations is complete.

The reduced propagator eliminates nonphysical repeated collisions in the spirit of multiple scattering expansion. If repeated collisions are not eliminated, which

is achieved using approximation  $G_{Q\downarrow} \approx G_{\downarrow}$ , one recovers the original Galitskii-Feynman approximation. Importance of the multiple scattering corrections can be seen from properties of the original Galitskii-Feynman approximation in the superconducting state. The T-matrix becomes singular which signals the onset of pairing. The single-particle propagator, however, does not have the gap in the energy spectrum.<sup>9,10</sup> With the multiple scattering corrections the gap develops.<sup>7,8</sup>

The half-selfconsistent theory<sup>10</sup> of Kadanoff and Martin (KM) is recovered if we approximate the  $Q$ -reduced propagator by the bare one,  $G_{Q\downarrow} \approx G^0$ . The KM theory yields the correct BCS gap but as noticed by Chen *et al*<sup>11</sup> and confirmed below, the KM approximation results in the ideal Bose gas of Cooper pairs. According to the Landau criterion the KM theory does not explain stability of the condensate with respect to excitation of noncondensed Cooper pairs. The reduced selfconsistency of the multiple scattering approach is thus essential for the excitation spectrum.

### B. Separable interaction

For a discussion in this paper we employ the simple BCS interaction

$$\begin{aligned} D(k, Q-k; p, Q-p) &= -V \theta(\omega_D - |\epsilon(k)|) \\ &\times \theta(\omega_D - |\epsilon(Q-k)|) \theta(\omega_D - |\epsilon(p)|) \theta(\omega_D - |\epsilon(Q-p)|). \end{aligned} \quad (7)$$

From Eq. (6) one can see that this separable potential implies the separable T-matrix

$$\begin{aligned} \mathcal{T}_{\uparrow\downarrow}(k, Q-k; p, Q-p) &= -\mathcal{T}_Q \theta(\omega_D - |\epsilon(k)|) \\ &\times \theta(\omega_D - |\epsilon(Q-k)|) \theta(\omega_D - |\epsilon(p)|) \theta(\omega_D - |\epsilon(Q-p)|) \end{aligned} \quad (8)$$

and Eq. (6) simplifies to a scalar equation

$$\frac{1}{\mathcal{T}_Q} = \frac{1}{V} + \frac{k_B T}{\Omega} \sum_k G_{\uparrow}(k) G_{Q\downarrow}(Q-k). \quad (9)$$

The sum over  $k$  is restricted by cutoffs of the BCS model. The  $Q$ -part of the selfenergy simplifies to

$$\sigma_{Q\uparrow}(k) = -\mathcal{T}_Q G_{Q\downarrow}(Q-k) \theta(\omega_D - |\epsilon(k)|). \quad (10)$$

Equations (1)-(3) and (9)-(10) form a closed set.

### C. Condensation mode

The condensation of Cooper pairs happens in a single mode. Below we prove this assumption. From now on we reserve the index  $Q$  for the condensation mode, while the other modes will be denoted by  $Q'$ . The Matsubara frequency in  $Q$  is zero,  $Q = (0, \mathbf{Q})$ .

The T-matrix of the  $Q$ -mode diverges reaching values proportional to the volume  $\Omega$ , see Ref. 7. To make a link

with the standard notation of the BCS theory we express this singular element as

$$\mathcal{T}_Q = \frac{\Omega}{k_B T} \bar{\Delta} \Delta, \quad (11)$$

and split the selfenergy into the singular contribution of the  $Q$ -mode and the regular reminder

$$\begin{aligned} \Sigma_{\uparrow}(k) &= -\bar{\Delta} G_{Q\downarrow}(-k) \Delta + \frac{k_B T}{\Omega} \sum_{Q' \neq Q} \sigma_{Q'\uparrow}(k) \\ &= -\bar{\Delta} G_{Q\downarrow}(-k) \Delta + \Sigma_{Q\uparrow}(k). \end{aligned} \quad (12)$$

According to Eqs. (1) and (3), the  $Q$ -reduced propagator relates to the full propagator as

$$G_{Q\downarrow} = G_{\downarrow} + G_{\downarrow} \bar{\Delta} G_{Q\uparrow} \Delta G_{Q\downarrow}. \quad (13)$$

If one neglects renormalizations keeping only the gap,  $\Sigma_{\uparrow} \approx -\bar{\Delta} G_{Q\downarrow} \Delta$ , i.e.,  $\Sigma_{Q\uparrow} \approx 0$  or  $G_{Q\downarrow} \approx G^0$ , this equation becomes identical to the Nambu-Gor'kov equation with the BCS gap  $\Delta$ .

The T-matrix of the  $Q$ -mode

$$\frac{1}{\mathcal{T}_Q} = \frac{1}{V} + \frac{k_B T}{\Omega} \sum_k G_{\uparrow}(k) G_{Q\downarrow}(Q-k) \quad (14)$$

determines the gap. In the thermodynamical limit  $\Omega \rightarrow \infty$ , the T-matrix of the condensation mode diverges, i.e.,  $1/\mathcal{T}_Q \rightarrow 0$ . Equation (14) then simplifies to the BCS-like gap equation

$$0 = \frac{1}{V} + \frac{k_B T}{\Omega} \sum_k G_{\uparrow}(k) G_{Q\downarrow}(Q-k). \quad (15)$$

Gor'kov have analyzed equation (15) close to the critical temperature  $T_c$ , where the gap is small. Keeping terms to the quadratic order in  $\Delta$  he has shown that it leads to

$$\frac{\mathbf{Q}^2}{2m^*} + \alpha + \beta |\Delta|^2 = 0, \quad (16)$$

where  $m^*$  is a Cooperon mass,  $\beta = 3/(2E_F)$  and  $\alpha = -6\pi^2 k_B^2 T_c (T_c - T) / (7\zeta_{[3]} E_F)$  are the GL parameters with  $n$  being the electron density,  $E_F$  the Fermi energy, and  $\zeta_{[3]} = 1.202$  the Riemann zeta function. We restrict our attention to the vicinity of the critical temperature. The limiting form (16) will be thus sufficient for our discussion.

#### D. Non-condensed Cooper pairs

We expect that none of terms for  $Q' \neq Q$  diverges with volume. This expectation is confirmed below. The  $Q'$ -reduced propagator then approaches the full one in the thermodynamical limit  $\Omega \rightarrow \infty$ ,

$$G_{Q'\uparrow} = G_{\uparrow}. \quad (17)$$

and the T-matrix of a  $Q' \neq Q$ -mode

$$\frac{1}{\mathcal{T}_{Q'}} = \frac{1}{V} + \frac{k_B T}{\Omega} \sum_k G_{\uparrow}(k) G_{\downarrow}(Q'-k) \quad (18)$$

thus satisfies equation distinct from Eq. (14). For the condensation mode (14) the gap enters only one of propagators while for noncondensation modes (18) both propagators depend on the gap. This difference results in a suppressed excitation of noncondensed Copper pairs.

The inverse T-matrix of noncondensation mode  $Q' = (0, \mathbf{Q}')$  results from expansion of Eq. (18) to the quadratic order in  $\Delta$  as

$$\frac{C}{\mathcal{T}_{Q'}} = \frac{\mathbf{Q}'^2}{2m^*} - |\alpha| + 2\beta |\Delta|^2, \quad (19)$$

where  $C = 8\pi^2 k_B^2 T_c^2 / (7\zeta_{[3]} n)$ . The factor of two in front of  $\beta$  follows from the fact that for noncondensed pairs both propagators depend on the gap.

### III. STABILITY OF THE CONDENSATE

Now we are ready to solve the central problem of this paper. First we show that once the condensate is formed in the  $Q$ -mode, a parallel condensation in another  $Q'$ -mode is excluded. Second we show that the critical velocity for breaking the condensed Cooper into two quasiparticles is lower than the critical velocity of excitation of Cooper pairs out of condensate.

#### A. Excluded parallel condensation

The inverse T-matrix (19) of the non-condensation  $Q'$ -mode cannot reach zero turning the  $Q'$ -mode into a parallel condensation mode. To show this we first use the GL equation (16), to express the T-matrix of noncondensation mode as

$$\frac{C}{\mathcal{T}_{Q'}} = \frac{\mathbf{Q}'^2}{2m^*} + |\alpha| - \frac{\mathbf{Q}^2}{m^*}. \quad (20)$$

Values of the pair momentum  $\mathbf{Q}$  are limited by the critical current,  $\mathbf{Q}^2 < \mathbf{Q}_c^2$ . The current is proportional to the square of the gap times the momentum,  $\mathbf{j} \propto \mathbf{Q} |\Delta|^2$ . Using Eq. (16) one finds  $\mathbf{j} \propto \mathbf{Q} (|\alpha| - \mathbf{Q}^2/2m^*)$ . The critical current is the maximum one,  $\partial \mathbf{j} / \partial \mathbf{Q} |_{\mathbf{Q}_c} = 0$ , which is achieved for  $\mathbf{Q}_c^2 = 2m^* |\alpha|/3$ , see Tinkham<sup>12</sup>. Accordingly,

$$\frac{C}{\mathcal{T}_{Q'}} > \frac{\mathbf{Q}'^2}{2m^*} + |\alpha| - \frac{\mathbf{Q}_c^2}{m^*} = \frac{\mathbf{Q}'^2}{2m^*} + \frac{|\alpha|}{3}. \quad (21)$$

Inequality (21) implies that the mode of  $Q' \neq Q$  cannot become singular once the condensation develops in the mode  $Q$ . Therefore, a parallel condensation in two competitive modes is excluded. Briefly, there is a single condensate, as it is tacitly assumed in the BCS theory.

## B. Excitation of Cooper pairs from the condensate

Now we discuss a possibility to excite a Cooper pair out of condensate by an object moving with velocity  $\mathbf{v}$  in the static condensate. Going into the floating coordinate system, this criterion is used to check stability of the condensate flowing with velocity  $-\mathbf{v}$  around a static obstacle.

The right hand side of Eq. (19) represents an energy of a noncondensed Cooper pair of momentum  $\mathbf{Q}'$ . In the frame floating with the condensate,  $\mathbf{Q} = \mathbf{0}$ , a Cooper pair can be excited from the condensate into a noncondensed state with the minimal energy cost  $|\alpha|$ . Let us estimate under which conditions Cooper pairs can be excited by an external perturbation.

According to the Landau criterion<sup>5</sup> the external perturbation moving with velocity  $\mathbf{v}$  can excite the Cooper pair of momentum  $\mathbf{Q}'$  if the Cherenkov condition

$$\mathbf{v}\mathbf{Q}' = \frac{\mathbf{Q}'^2}{2m^*} + |\alpha| \quad (22)$$

is satisfied. This equation is solved by real  $\mathbf{Q}'$  for

$$|\mathbf{v}| > \sqrt{\frac{2|\alpha|}{m^*}}. \quad (23)$$

This velocity is higher than the critical velocity of pair breaking  $v_c = \Delta/k_F$ , where  $k_F$  is the Fermi momentum. Indeed, from Eq. (16) follows  $\Delta = \sqrt{|\alpha|/\beta} = \sqrt{|\alpha|k_F^2/(3m)}$ , where  $m = m^*/2$  is the electronic mass, therefore

$$|\mathbf{v}| > \sqrt{3}v_c. \quad (24)$$

Briefly, it is easier to break a Cooper pair into two quasi-particles than to excite it from the condensate into a noncondensed Cooper pair.

## C. Role of the multiple scattering corrections

As pointed out above deriving Eqs. (14) and (18), within the multiple scattering approach the propagators inside the T-matrix depend on the evaluated scattering process. As the condensation mode becomes singular, the two-particle propagations in the condensed and noncondensed modes become particularly different. Let us

show that this difference is essential using the Kadanoff-Martin approximation which results from the present approximation using  $G_{Q\downarrow} \approx G^0$ , therefore it uses the same two-particle propagation  $G_{\uparrow}(k)G^0(Q-k)$  for all modes.

Now we confirm that the KM approximation results in the ideal gas of Cooper pairs<sup>11</sup>. Since one of propagators is bare for all modes, the KM counterpart of equation (19) reads  $\frac{C}{\tau_{Q'}} = \frac{\mathbf{Q}'^2}{2m^*} + |\alpha| + \beta|\Delta|^2$  so that equation (20) modifies to  $\frac{C}{\tau_{Q'}} = \frac{\mathbf{Q}'^2}{2m^*} - \frac{\mathbf{Q}^2}{2m^*}$ . When the condensate moves,  $\mathbf{Q} \neq 0$ , it is energetically favorable to start condensation in standing mode  $\mathbf{Q}' = 0$  which stops the supercurrent. A similar problem appears for noncondensed Cooper pairs. In the frame moving with the condensate one finds the free-particle-like energy of noncondensed Cooper pairs  $\mathbf{Q}'^2/2m^*$ . Therefore, according to the Landau criterion in the KM approximation the condensate is not stable.

## IV. SUMMARY

We have discussed stability of supercurrents with respect to condensation in competitive modes and excitations of noncondensed Cooper pairs. It was shown within the Galitskii-Feynman approximation that multiple scattering corrections yield the familiar BCS gap in the single-particle energy spectrum and also a smaller gap in the two-particle energy spectrum separating the noncondensed Cooper pairs from the condensate. This two-particle gap prevents parallel condensation of Cooper pairs in two or more modes. Moreover, due to the two-particle gap the critical velocity to excite noncondensed Cooper pairs is higher than the critical velocity of the pair breaking, therefore the noncondensed Cooper pairs do not affect stability of supercurrents. The present result justifies basic assumptions of the BCS theory in which the condensate is expected in a single mode and Cooper pairs out of the condensate are ignored.

## Acknowledgments

This work was supported by research plans MSM 0021620834, grants GAUK 135909, GAČR 204/10/0687 and 202/08/0326, GAAV 100100712, DAAD PPP, and by DGF-CNPq project 444BRA-113/57/0-1.

<sup>1</sup> L. P. Gor'kov, Zh. Eksper. Teor. Fiz. **36**, 1918 (1959), [Sov. Phys. JETP **9**, 1364 (1959)].

<sup>2</sup> J. Bardeen, L. Cooper, and J. Schrieffer, Phys. Rev. **108**, 1175 (1957).

<sup>3</sup> J. Goldstone, Nuovo Cim. **19**, 154 (1961).

<sup>4</sup> H. Shi and A. Griffin, Phys. Rep. **304**, 1 (1998).

<sup>5</sup> C. J. Pethick and H. Smith, *Bose-Einstein Condensa-*

*tion in Dilute Gases* (Cambridge University Press, Oxford, 2001).

<sup>6</sup> J. A. Pals, K. Weiss, P. M. T. M. Vanattekum, R. E. Horstman, and J. Wolter, Phys. Rep. **89**, 324 (1982).

<sup>7</sup> P. Lipavský, Phys. Rev. B **78**, 214506 (2008).

<sup>8</sup> B. Šopík, P. Lipavský, M. Männel, and K. Morawetz, arXiv:0906.3677v2 (2010).

- <sup>9</sup> W. Wild, Z. Phys. A **158**, 322 (1960).
- <sup>10</sup> L. P. Kadanoff and P. C. Martin, Phys. Rev. **124**, 670 (1961).
- <sup>11</sup> Q. Chen, J. Stajic, S. Tan, and K. Levin, Phys. Rep. **412**, 1 (2005).
- <sup>12</sup> M. Tinkham, *Introduction to Superconductivity* (McGraw Hill, New York, 1966).